DUALITY FOR LIPSCHITZ p-SUMMING OPERATORS

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Abstract. Building upon the ideas of R. Arens and J. Eells [AE56] we introduce the concept of spaces of Banach-space-valued molecules, whose duals can be naturally identified with spaces of operators between a metric space and a Banach space. On these spaces we define analogues of the tensor norms of Chevet and Saphar [Che69, Sap70], whose duals are spaces of Lipschitz p-summing operators. In particular, we identify the dual of the space of Lipschitz p-summing operators from a finite metric space to a Banach space — answering a question of J. Farmer and W. B. Johnson [FJ09] — and use it to give a new characterization of the non-linear concept of Lipschitz p-summing operator between metric spaces in terms of linear operators between certain Banach spaces. More generally, we define analogues of the norms of J.T. Laprest́e [Lap76], whose duals are analogues of A. Pietsch's (p, r, s)-summing operators [Pie80]. As a special case, we get a Lipschitz version of (q, p)-dominated operators.

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1. Introduction

The origin of this work was the following question: given a normed space of Lipschitz maps from a metric space X into a Banach space E (e.g. Lipschitz p-summing operators as in [FJ09]) how can one identify its dual?

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A natural starting point would be to try to identify the dual of $X^{\#}$, the space of Lipschitz functions from X to $\mathbb{R}$ that vanish at a specified point with the Lipschitz norm. Unfortunately, duals of spaces of Lipschitz functions are known to be rather large and somewhat pathological — e.g. in [Bou86] it is shown that $(\ell_1)^{\#}$ does not have finite cotype, and it is still unknown whether $([0,1]\times[0,1])^{\#}$ has finite cotype —, so ours would appear to be a futile task.

We may, however, flip the table and get back into a workable situation: the space $X^{\#}$ is known to be a dual Banach space (and is sometimes even called the Lipschitz dual of X), so we embark on the slightly different (but related) quest of finding preduals of some spaces of Lipschitz maps from a metric space X into a dual Banach space E^*.

The key element in our work is the concept of a Banach-space-valued molecule, a generalization of the concept used by R. Arens and J. Eells [AE56] to construct a predual of $X^{\#}$. Despite the fact that the Arens-Eells space has been used repeatedly in the literature (e.g. [GK03, Kal04]), and Banach-space valued versions of it have been considered (as in [Joh70]), as far as this author knows the idea of Banach-space-valued molecules had escaped attention so far.

The rest of the paper is organized as follows. In section 2 we fix our notation and recall the definition and some basic properties of Lipschitz p-summing operators. In section 3 we revisit the construction of the Arens-Eells space [AE56] and introduce the Banach-space-valued version of their concept of molecules. Next, in section 4, we define norms on the spaces of molecules that are inspired by the tensor norms introduced by S. Chevet and P. Saphar [Che69, Sap70] and show how the resulting dual spaces are precisely the spaces of Lipschitz p-summing operators. Furthermore, we use this duality to prove a characterization of Lipschitz p-summing operators between metric spaces via induced mappings between spaces of molecules. Finally, in section 5 we introduce more general norms for the spaces of molecules modeled after the tensor norms of J.T. Lapresté [Lap76], and study their duality, arriving at the concepts of Lipschitz (p,r,s)-summing operators and Lipschitz (q,p)-dominated operators.

2. Notation and preliminaries

X, Y, Z will always denote metric spaces, whereas E, F, G will denote real Banach spaces. We use the convention of having *pointed* metric spaces, i.e. with a designated special point always denoted by 0. As customary, B_E denotes the closed unit ball of E and E^* its linear dual, and $\mathcal{L}(E,F)$ is the space of bounded linear maps from E to F. We use the symbol $\equiv$ to indicate that two Banach spaces are isometrically isomorphic. $\mathrm{Lip}_0(X,E)$ is the Banach space of Lipschitz functions $T : X \to E$ such that $T(0) = 0$ with pointwise addition and the Lipschitz norm. As the reader will recall from the introduction, we use the shorthand $X^{\#} := \mathrm{Lip}_0(X,\mathbb{R})$. The letters p, r, s will designate elements of $[1,\infty]$, and p' denotes the exponent conjugate to p (i.e. the one that satisfies $1/p + 1/p' = 1$).

Absolutely summing operators are by now widely recognized as one of the most important developments in modern Banach space theory, as attested to by the astonishing number of results and applications that can be found, for example, in [DJT95]. Let us recall that for $1 \leq p < \infty$, a linear map $T : E \to F$ is p-summing if there exists a constant $C \geq 0$ such that regardless of the choice of vectors $v_1 \dots, v_n$ in E we have

$$\left[\sum_{j=1}^{n} \|Tv_j\|^p\right]^{1/p} \leq C \sup_{v^* \in B_{X^*}} \left[\sum_{j=1}^{n} |v^*(v_j)|^p\right]^{1/p}.$$

The infimum of such constants C is denoted by $\pi_p(T)$ and called the p-summing norm of T. Inspired by this useful concept, J. Farmer and W. B. Johnson introduced in [FJ09] the following definition: a Lipschitz map $T : X \to Y$ is called *Lipschitz p-summing* if there exists a constant $C \geq 0$ such that regardless of the choice of points $x_1, \dots, x_n, x'_1, \dots, x'_n$ in X we have the inequality

$$\left[\sum_{j=1}^{n} d(Tx_j, Tx'_j)^p\right]^{1/p} \leq C \sup_{f \in B_{X^*}} \left[\sum_{j=1}^{n} \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}.$$

The infimum of such constants is denoted by $\pi_p^L(T)$. This is a true generalization of the concept of linear p-summing operator, since it is shown in [FJ09, Thm. 2] that the Lipschitz p-summing norm of a linear operator is the same as its p-summing norm.

In order to shorten the notation and avoid having to treat the case $p = \infty$ separately, we introduce some more symbols and terminology. $\|\cdot\|_p$ denotes the norm on ℓ_p of a sequence of real numbers. All sequences (of numbers and vectors) under consideration in this paper will be finite, so there will be no issues of convergence. For a sequence of vectors $(v_j)_j$ in a Banach space E, its *strong p-norm* is the ℓ_p-norm of the sequence $(\|v_j\|)_j$ and we denote its *weak p-norm* (cf. [DF93, p. 91]) by

$$w_p\big((v_j)_j\big) := \sup_{v^* \in B_{E^*}} \left\| \big(v^*(v_j)\big)_j \right\|_p.$$

Analogously, for sequences of the same length $(\lambda_j)_j$ of real numbers and $(x_j)_j$, $(x'_j)_j$ of points in X, we denote their *weak Lipschitz p-norm* by

$$w_p^{\mathrm{Lip}}\big((\lambda_j, x_j, x'_j)_j\big) := \sup_{f \in B_{X^\#}} \left\| \big(\lambda_j [f(x_j) - f(x'_j)]\big)_j \right\|_p.$$

3. Banach-space-valued molecules on a metric space

We start by recalling the definition and basic properties of the space of Arens and Eells [AE56]. We follow the presentation in [Wea99].

A *molecule* on a metric space X is a finitely supported function $m : X \to \mathbb{R}$ such that $\sum_{x \in X} m(x) = 0$. The simplest molecules, i.e. those of the form $m_{xx'} := \chi_{\{x\}} - \chi_{\{x'\}}$ with $x, x' \in X$ are called *atoms*. The Arens-Eells space of X, denoted Æ(X), is the completion of the space of molecules with the norm

$$\|m\|_{Æ} := \inf\left\{ \sum_{j=1}^{n} |a_j| d(x_j, x'_j) \; : \; m = \sum_{j=1}^{n} a_j m_{x_j x'_j} \right\}. \tag{3.1}$$

The fundamental properties of the Arens-Eells space are summarized in the following theorem [AE56], [Wea99, pp. 39-41].

Theorem 3.1. *(i)* $\|\cdot\|_{Æ}$ *is a norm on the vector space of molecules on* X*.*

(ii) The dual of $Æ(X)$ *is (canonically) isometrically isomorphic to* $X^{\#}$*. Moreover, on bounded subsets of* $X^{\#}$ *the weak* topology coincides with the topology of pointwise convergence.*

(iii) The map $\iota : x \mapsto m_{x0}$ *is an isometric embedding of* X *into* $Æ(X)$*. Moreover, for any Banach space* E *and any Lipschitz map* $T : X \to E$ *with* $T(0) = 0$ *there is a unique linear map* $\hat{T} : Æ(X) \to E$ *such that* $\hat{T} \circ \iota = T$*. Furthermore,* $\|\hat{T}\| = \mathrm{Lip}(T)$*.*

Because of the universal property (iii), the space $Æ(X)$ is sometimes called the free Lipschitz space of X (see [GK03, Kal04]). From that point of view, it is natural to think of the space $Æ(X)$ as the closure in $(X^{\#})^*$ of the linear span of the point evaluations $\delta_x : f \mapsto f(x)$, for $x \in X$ and $f \in X^{\#}$. Such an approach was used by J. Johnson [Joh70] to show that $\mathrm{Lip}_0(X, E^*)$ is always a dual space, without any reference to molecules. Our Theorem 4.3 recovers Johnson's result as a particular case of duality for Lipschitz p-summing operators.

In the spirit of Arens and Eells original formulation [AE56], define an *E-valued molecule* on X to be a finitely supported function $m : X \to E$ such that $\sum_{x\in X} m(x) = 0$. The vector space of all E-valued molecules on X is denoted by $\mathcal{M}(X, E)$. An *E-valued atom* is a function of the form $vm_{xx'}$ with $v \in E$, $x, x' \in X$. Atoms are the building blocks of the space of molecules in the same sense that elementary tensors are the building blocks of the tensor product: every molecule is a sum of atoms. This is proved by induction on the cardinality of the support of the molecule as follows. It is clear if the support has cardinality 0 or 2 (1 is clearly impossible), so now suppose the result holds for molecules with support of size at most n, and let $m : X \to E$ be a molecule with support $\{x_0, x_1, \dots, x_n\} \subset X$. Note that $\tilde{m} = m - \sum_{j=1}^{n} \frac{1}{n} m(x_0) m_{x_0 x_j}$ is a molecule with support of size at most n (since $\tilde{m}(x_0) = 0$), so $\tilde{m}$ is a sum of atoms and therefore clearly so is m.

Define a pairing $\langle \cdot, \cdot \rangle$ of $\mathrm{Lip}_0(X, E^*)$ and $\mathcal{M}(X, E)$ by

$$\langle T, m\rangle = \sum_{x\in X} \langle T(x), m(x)\rangle \qquad m \in \mathcal{M}(X, E), \quad T \in \mathrm{Lip}_0(X, E^*).$$

Note that this sum makes sense because m is finitely supported, and clearly $\langle \cdot, \cdot \rangle$ is bilinear. For an atom $m = vm_{x'y'}$ and $T \in \mathrm{Lip}_0(X, E^*)$,

$$\langle T, m\rangle = \sum_{x\in X} \langle T(x), vm_{x'y'}(x)\rangle = \langle T(x'), vm_{x'y'}(x')\rangle + \langle T(y'), vm_{x'y'}(y')\rangle = \langle T(x') - T(y'), v\rangle.$$

Therefore, for a general molecule $m = \sum_j v_j m_{x_j x'_j}$,

$$\langle T, m\rangle = \sum_{j} \langle Tx_j - Tx'_j, v_j\rangle. \tag{3.2}$$

4. The Chevet-Saphar norms

When the tensor product $E \otimes F$ of two Banach spaces is endowed with a tensor norm, its dual space can be interpreted as linear operators from E to F^*. Under (some of) the Chevet-Saphar tensor norms, introduced independently by S. Chevet [Che69] and P. Saphar [Sap70] as generalizations of earlier work of Saphar [Sap65], the operators from E to F^* obtained in this way are precisely p-summing operators. The main result of this section (Theorem 4.3) is the analogous result in the setting of Lipschitz p-summing operators between a metric space and a Banach space, with the space of molecules playing the role of the tensor product in the linear theory.

4.1. **Definition and elementary properties.** For a molecule $m \in \mathcal{M}(X, E)$ we define its *p-Chevet-Saphar norm* by

$$cs_p(m) = \inf\left\{ \left\| \big(\lambda_j \left\|v_j\right\| \big)_j \right\|_p w_{p'}^{\mathrm{Lip}}\big((\lambda_j^{-1}, x_j, x_j')_j\big) \,:\, m = \sum_j v_j m_{x_j x_j'} \right\}. \tag{4.1}$$

The reader familiar with the theory of Chevet-Saphar norms on tensor products of Banach spaces will recall that there are two versions of those norms for a given index p; a *left* one and a *right* one. Such variants are also possible in the present context, but we stick with only one for now and postpone the study of the other one until section 5, when we tackle the more general Lapresté norms. Let us start by showing that our use of the word "norm" is justified.

Theorem 4.1. *cs_p is a norm on $\mathcal{M}(X, E)$.*

Proof. It is clear that for any molecule $m \in \mathcal{M}(X, E)$ and any scalar λ, $cs_p(m) \geq 0$ and $cs_p(\lambda m) = |\lambda| cs_p(m)$. Let $m_1, m_2 \in \mathcal{M}(X, E)$ and $\varepsilon > 0$. By definition of the cs_p norm we can find a representation $m_1 = \sum_j v_j m_{x_j x_j'}$ and a sequence of positive reals $(\lambda_j)_j$ such that

$$\left\| \big(\lambda_j \left\|v_j\right\| \big)_j \right\|_p w_{p'}^{\mathrm{Lip}}\big((\lambda_j^{-1}, x_j, x_j')_j\big) \leq cs_p(m_1) + \varepsilon.$$

Replacing $(\lambda_j)_j$ by an appropriate multiple of it, we may in fact assume that

$$\left\| \big(\lambda_j \left\|v_j\right\| \big)_j \right\|_p \leq \big(cs_p(m_1) + \varepsilon\big)^{1/p}, \qquad w_{p'}^{\mathrm{Lip}}\big((\lambda_j^{-1}, x_j, x_j')_j\big) \leq \big(cs_p(m_1) + \varepsilon\big)^{1/p'}. \tag{4.2}$$

Similarly, there exist a representation $m_2 = \sum_i w_i m_{y_i y_i'}$, and positive reals $(\kappa_i)_i$ such that

$$\left\| \big(\kappa_i \left\|w_i\right\| \big)_i \right\|_p \leq \big(cs_p(m_2) + \varepsilon\big)^{1/p}, \qquad w_{p'}^{\mathrm{Lip}}\big((\kappa_i^{-1}, y_i, y_i')_i\big) \leq \big(cs_p(m_2) + \varepsilon\big)^{1/p'}. \tag{4.3}$$

We now "glue" together these representations of m_1 and m_2 to get a representation of $m_1 + m_2$: let $(u_k)_k$ be the sequence obtained from concatenating $(v_j)_j$ and $(w_i)_i$; similarly obtain $(z_k, z_k')_k$ from concatenating $(x_j, x_j')_j$ and (y_i, y_i'); and construct $(\eta_k)_k$ from $(\lambda_j)_j$ and $(\kappa_i)_i$. Then the strong p-norm of $(\eta_k u_k)_k$ is just the p-sum of the strong p-norms of $(\lambda_j v_j)_j$ and $(\kappa_i w_i)_i$, so from (4.2) and (4.3) we have

$$\left\| \big(\eta_k \left\|u_k\right\| \big)_k \right\|_p \leq \big(cs_p(m_1) + cs_p(m_2) + 2\varepsilon\big)^{1/p}. \tag{4.4}$$

Similarly, the weak Lipschitz p'-norm of $(\eta_k^{-1}, z_k, z'_k)_k$ is bounded above by the p'-sum of the weak Lipschitz p'-norms of $(\lambda_j^{-1}, x_j, x'_j)_j$ and $(\kappa_i^{-1}, y_i, y'_i)$, so once more from (4.2) and (4.3) we obtain

$$w_{p'}\big((\eta_k^{-1}, z_k, z'_k)_k\big) \leq \big(cs_p(m_1) + cs_p(m_2) + 2\varepsilon\big)^{1/p'}. \tag{4.5}$$

But clearly $m_1 + m_2 = \sum_k u_k m_{z_k z'_k}$, so the product of (4.4) and (4.5) together with the definition of cs_p give $cs_p(m_1+m_2) \leq cs_p(m_1) + cs_p(m_2) + 2\varepsilon$. By letting ε tend to zero we have the triangle inequality for cs_p.

Let $T \in \mathrm{Lip}_0(X, E^*)$ be a map that admits a representation as a finite sum of the form $\sum_k v_k^* f_k$ with $(v_k^*)_k \subset E^*$, $(f_k)_k \subset X^\#$ (i.e. such that the linearization $\hat{T} : \text{Æ}(X) \to E^*$ has finite rank). For such a T, set

$$\theta_p(T) = \inf\Big\{ \big\| \big(\|v_k^*\| \big)_k \big\|_p \big\| \big(\mathrm{Lip}(f_k) \big)_k \big\|_{p'} \Big\}$$

where the infimum is taken over all representations as above. Now, given $m = \sum_j v_j m_{x_j x'_j} \in \mathcal{M}(X, E)$, and $(\lambda_j)_j$ a sequence of positive real numbers, from the pairing formula (3.2) and Hölder's inequality

$$\begin{aligned} \big|\langle T, m\rangle\big| = \Big| \sum_{j,k} v_k^*(v_j)\big[f_k(x_j) - f_k(x'_j)\big] \Big| &\leq \sum_{j,k} \Big| \lambda_j v_k^*(v_j) \lambda_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big] \Big| \\ &\leq \Big\| \big(\lambda_j v_k^*(v_j)\big)_{j,k} \Big\|_p \Big\| \Big(\lambda_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big] \Big)_{j,k} \Big\|_{p'}. \end{aligned} \tag{4.6}$$

For finite p, the definition of the ℓ_p-norm gives

$$\Big\| \big(\lambda_j v_k^*(v_j)\big)_{j,k} \Big\|_p^p = \sum_k \sum_j |\lambda_j|^p |v_k^*(v_j)|^p \leq \sum_k \|v_k^*\|^p \sum_j |\lambda_j|^p \, \|v_j\|^p$$

so after taking the p-th root

$$\Big\| \big(\lambda_j v_k^*(v_j)\big)_{j,k} \Big\|_p \leq \Big\| \big(\lambda_j \, \|v_j\| \,\big)_j \Big\|_p \big\| \big(\|v_k^*\| \big)_k \big\|_p \tag{4.7}$$

and the same inequality is also trivially valid for $p = \infty$. On the other hand, by an analogous argument,

$$\Big\| \Big(\lambda_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big] \Big)_{j,k} \Big\|_{p'} \leq \big\| \big(\mathrm{Lip}(f_k) \big)_k \big\|_{p'} \, w_{p'}^{\mathrm{Lip}}\Big(\big(\lambda_j^{-1}, x_j, x'_j\big)_j \Big). \tag{4.8}$$

Together, equations (4.6), (4.7) and (4.8) imply

$$\big|\langle T, m\rangle\big| \leq \Big\| \big(\lambda_j \, \|v_j\| \,\big)_j \Big\|_p \, w_{p'}^{\mathrm{Lip}}\Big(\big(\lambda_j^{-1}, x_j, x'_j\big)_j \Big) \, \big\| \big(\|v_k^*\| \big)_k \big\|_p \big\| \big(\mathrm{Lip}(f_k) \big)_k \big\|_{p'}$$

so after taking the infimum over all representations, $\big|\langle T, m\rangle\big| \leq cs_p(m)\theta_p(T)$. In particular, this applies to maps T of the form $v^* f$ with $v^* \in E^*$ and $f \in X^\#$, so if m is such that $cs_p(m) = 0$ then we have, using the pairing formula (3.2),

$$0 = \langle v^* f, m\rangle = \sum_j v^*(v_j)\big[f(x_j) - f(x'_j)\big] \text{ for all } v^* \in E^*, f \in X^\#.$$

By the duality between Æ(X) and $X^\#$ (see Theorem 3.1), this means that the real-valued molecule $v^* \circ m$ is equal to 0 for all $v^* \in E^*$ and consequently $m = 0$. □

We will denote by $\mathcal{CS}_p(X, E)$ the normed space $\big(\mathcal{M}(X, E), cs_p\big)$. Notice that when X is a finite set the space $\mathcal{CS}_p(X, E)$ is complete, since it is isomorphic to $E^{|X|-1}$. On the other hand, when the set X is infinite the elements of the completion of $\mathcal{CS}_p(X, E)$ correspond to infinite representations as sums of atoms that are analogous to the ones considered in (4.1), but we need not concern ourselves with such technicalities for our present purposes.

The next proposition shows that in the extreme cases $p = 1$ and $p = \infty$, cs_p can be calculated using a simpler formula. In particular, we obtain that the cs_1 norm is just the straightforward generalization of the Arens-Eells norm to the Banach-valued case (cf. (3.1)).

Proposition 4.2. *For a molecule $m \in \mathcal{M}(X, E)$,*

$$cs_1(m) = \inf\left\{ \sum_j \|v_j\|\, d(x_j, x'_j) : m = \sum_j v_j m_{x_j x'_j} \right\} \tag{4.9}$$

and

$$cs_\infty(m) = \inf\left\{ \sup_{f \in B_{X^\#}} \sum_j \|v_j\| \left|f(x_j) - f(x'_j)\right| : m = \sum_j v_j m_{x_j x'_j} \right\} \tag{4.10}$$

Proof. Start by noting that given positive numbers λ_j and points x_j, x'_j in X,

$$w_\infty^{\mathrm{Lip}}\big((\lambda_j, x_j, x'_j)_j\big) = \sup_{f \in B_{X^\#}} \left\| \big(\lambda_j [f(x_j) - f(x'_j)]\big)_j \right\|_\infty = \max_j \lambda_j d(x_j, x'_j), \tag{4.11}$$

because for each j, $\big|f(x_j) - f(x'_j)\big|$ is at most $d(x_j, x'_j)$ whenever $f \in B_{X^\#}$ and this upper bound is in fact achieved: given any two points $x, x' \in X$, the function $f : X \to \mathbb{R}$ given by $f(\cdot) = d(\cdot, x') - d(x', 0)$ is in $\mathrm{Lip}_0(X, \mathbb{R})$, has Lipschitz constant 1 and satisfies $\big|f(x) - f(x')\big| = d(x, x')$. Now, given a molecule $m = \sum_j v_j m_{x_j y_j}$ and positive reals $(\lambda_j)_j$, (4.11) gives

$$\begin{aligned} \left\| \big(\lambda_j \|v_j\|\big)_j \right\|_1 w_\infty^{\mathrm{Lip}}\big((\lambda_j^{-1}, x_j, x'_j)\big) &= \Big(\sum_j \lambda_j \|v_j\|\Big)\Big(\max_j \lambda_j^{-1} d(x_j, x'_j)\Big) \\ &\geq \sum_j \lambda_j^{-1} d(x_j, x'_j) \lambda_j \|v_j\| = \sum_{j=1}^n \|v_j\|\, d(x_j, x'_j). \end{aligned}$$

Taking the infimum over all representations of m we get the inequality $\geq$ in (4.9). On the other hand, (4.11) with the particular choice $\lambda_j = d(x_j, x'_j)$ gives

$$\begin{aligned} cs_1(m) &\leq \left\| \big(d(x_j, x'_j) \|v_j\|\big)_j \right\|_1 w_\infty^{\mathrm{Lip}}\big((d(x_j, x'_j)^{-1}, x_j, x'_j\big)_j \\ &= \Big(\sum_j d(x_j, x'_j) \|v_j\|\Big) \max_j \frac{d(x_j, x'_j)}{d(x_j, x'_j)} = \sum_j d(x_j, x'_j) \|v_j\| \end{aligned}$$

and after taking the infimum over all representations of m we obtain $\leq$ in (4.9).

Now, given a molecule $m = \sum_j v_j m_{x_j x'_j}$ and positive numbers λ_j,

$$\Big\| \big(\lambda_j \, \|v_j\| \big)_j \Big\|_\infty w_1^{\mathrm{Lip}}\big((\lambda_j^{-1}, x_j, x'_j) \big) = \Big(\max_j \lambda_j \, \|v_j\| \Big) \sup_{f \in B_{X^\#}} \sum_j \lambda_j^{-1} \big| f(x_j) - f(x'_j) \big|$$

$$\geq \sup_{f \in B_{X^\#}} \sum_j \lambda_j \, \|v_j\| \, \lambda_j^{-1} \big| f(x_j) - f(x'_j) \big| = \sup_{f \in B_{X^\#}} \sum_j \|v_j\| \, \big| f(x_j) - f(x'_j) \big|$$

so taking the infimum over all representations gives $\geq$ in (4.10). On the other hand,

$$cs_\infty(m) \leq \Big\| \big(\, \|v_j\|^{-1} \, \|v_j\| \, \big)_j \Big\|_\infty w_1^{\mathrm{Lip}}\big((\|v_j\|, x_j, x'_j) \big) = 1 \cdot \sup_{f \in B_{X^\#}} \sum_j \|v_j\| \, \big| f(x_j) - f(x'_j) \big|$$

and taking the infimum yet again rewards us with $\leq$ in (4.10). □

4.2. **An example.** We now use Proposition 4.2 to calculate explicitly the space $\mathcal{CS}_1$ in the case when X is a graph-theoretic tree. First note that (4.9) can be interpreted as saying that in general the space $\mathcal{CS}_1(X, E)$ is a quotient of a weighted (with weight given by the distance d) ℓ_1-sum of copies of E; in fact the quotient map $Q : \big(\bigoplus_{x, x' \in X} E \big)_{\ell_1, d} \to \mathcal{M}(X, E)$ is given by

$$Q\big((v_{xx'})_{x, x' \in X} \big) = \sum_{x, x' \in X} v_{xx'} m_{xx'}.$$

When X is a graph-theoretic tree, we will show that $\mathcal{CS}_1(X, E)$ is again a weighted ℓ_1-sum of copies of E. Let $\mathcal{T} = (\mathcal{V}, \mathcal{E}, w)$ be a graph-theoretic weighted tree with vertex set $\mathcal{V}$, edge set $\mathcal{E}$ and weight function $w : \mathcal{E} \to \mathbb{R}^+$, considered as a pointed metric space. Edges in $\mathcal{E}$ will be written as ordered pairs (u, v) with u closer to 0 than v.

Consider $\mathsf{u}, \mathsf{v} \in \mathcal{V}$. Let $\{\mathsf{u} = \mathsf{w}_0, \mathsf{w}_1, \dots, \mathsf{w}_n = \mathsf{v}\}$ be the unique minimal-length path joining u and v. Since

$$d(\mathsf{u}, \mathsf{v}) = \sum_{i=1}^n d(\mathsf{w}_i, \mathsf{w}_{i-1}),$$

a look at equation (4.9) reveals that in order to calculate the cs_1 norm of a molecule it suffices to consider only representations involving atoms m_{vu} with $(\mathsf{u}, \mathsf{v}) \in \mathcal{E}$. By the triangle inequality in E, for the purpose of calculating the cs_1 norm of a molecule m in its representation we can consolidate all terms corresponding to the same atom m_{uv}, so in conclusion we can consider only representations of the form

$$m = \sum_{(\mathsf{u}, \mathsf{v}) \in \mathcal{E}} v_{\mathsf{vu}} m_{\mathsf{vu}} \qquad \text{with } v_{\mathsf{vu}} \in E. \tag{4.12}$$

But for a given molecule there is only one such representation (recall we have fixed an orientation for every edge), a fact easily proved by induction on the size of the tree, so the cs_1 norm of a molecule m is the weighted (with weight w) ℓ_1-norm of the associated vector $(y_{\mathsf{vu}})_{(\mathsf{u}, \mathsf{v}) \in \mathcal{E}}$ given by the unique representation from (4.12). Therefore, the $\mathcal{CS}_1$ space of a graph-theoretic weighted tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}, w)$ is the weighted ℓ_1-sum of copies of E indexed by $\mathcal{E}$ and with weight w.

4.3. **Duality.** We show now that the duals of the Chevet-Saphar spaces of molecules can be canonically identified as spaces of Lipschitz p-summing operators.

Theorem 4.3. *The spaces $\mathcal{CS}_p(X,E)^*$ and $\Pi^L_{p'}(X,E^*)$ are isometrically isomorphic via the canonical pairing. Moreover, on the unit ball of $\Pi^L_{p'}(X,E^*)$ the weak* topology coincides with the topology of pointwise $\sigma(E^*,E)$-convergence.*

Proof. First, let $T \in \Pi^L_{p'}(X,E^*)$. Consider a molecule $m = \sum_j v_j m_{x_j x'_j} \in \mathcal{M}(X,E)$ and positive numbers λ_j. The pairing formula 3.2, Hölder's inequality and the definition of Lipschitz p'-summing naturally come together to give us

$$\begin{aligned}\left|\langle T,m\rangle\right| &= \left|\sum_j \langle Tx_j - Tx'_j, v_j\rangle\right| \le \sum_j \left|\langle Tx_j - Tx'_j, v_j\rangle\right| \le \sum_j \left\|Tx_j - Tx'_j\right\| \|v_j\| \\ &\le \|(\lambda_j\|v_j\|)_j\|_p \left\|\left(\lambda_j^{-1}\left\|Tx_j - Tx'_j\right\|\right)_j\right\|_{p'} \le \|(\lambda_j\|v_j\|)_j\|_p\, \pi^L_{p'}(T) w^{\mathrm{Lip}}_{p'}\big((\lambda_j^{-1}, x_j, x'_j)_j\big).\end{aligned}$$

Taking the infimum over all representations of m and positive λ_j we conclude that

$$\left|\langle T,m\rangle\right| \le \pi^L_{p'}(T) cs_p(m).$$

Conversely, let $\varphi \in \mathcal{CS}_p(X,E)^*$ with $\|\varphi\| = C$. Then we have $|\varphi(m)| \le C cs_p(m)$ for any $m \in \mathcal{M}(X,E)$. Note that φ can be identified with a mapping $T : X \mapsto E^*$ via the formula $\langle Tx, v\rangle = \varphi(v m_{x0})$. Indeed, for $x \in X$ and $v \in E$,

$$|\langle Tx,v\rangle| = |\langle \varphi, v m_{x0}\rangle| \le C cs_p(v m_{x0}) \le C\|v\| \sup_{f\in B_{X^\#}} |f(x) - f(0)| = C\|v\| d(x,0)$$

so $Tx \in E^*$. Now, fix points x_j, x'_j in X and positive numbers λ_j, $j = 1,\dots,n$. Let $\varepsilon > 0$. For each j pick $v_j \in E$ such that $\langle Tx_j - Tx'_j, v_j\rangle = \left\|Tx_j - Tx'_j\right\|$ and $\|v_j\| \le 1+\varepsilon$. Then, for any sequence $(\alpha_j)_j$ of real numbers,

$$\begin{aligned}\left|\sum_j \alpha_j\lambda_j\langle Tx_j - Tx'_j, v_j\rangle\right| &= \left|\langle T, \sum_j \alpha_j\lambda_j v_j m_{x_j x'_j}\rangle\right| \le C cs_p\Big(\sum_j \alpha_j\lambda_j v_j m_{x_j x'_j}\Big) \\ &\le C\|(\alpha_j\|v_j\|)_j\|_p\, w^{\mathrm{Lip}}_{p'}\big(\lambda_j, x_j, x'_j\big) \le C(1+\varepsilon)\|(\alpha_j)_j\|_p\, w^{\mathrm{Lip}}_{p'}\big(\lambda_j, x_j, x'_j\big).\end{aligned}$$

Taking the supremum over all sequences with $\|(\alpha_j)_j\|_p \le 1$,

$$\left\|\big(\lambda_j\langle Tx_j - Tx'_j, v_j\rangle\big)_j\right\|_{p'} \le C(1+\varepsilon) w^{\mathrm{Lip}}_{p'}\big(\lambda_j, x_j, x'_j\big).$$

Letting ε go to 0,

$$\left\|\big(\lambda_j\left\|Tx_j - Tx'_j\right\|\big)_j\right\|_{p'} \le C w^{\mathrm{Lip}}_{p'}\big(\lambda_j, x_j, x'_j\big),$$

i.e. T is Lipschitz p'-summing with $\pi^L_{p'}(T) \le C$.

For the second part, suppose $(T_\alpha)_\alpha \subset \Pi^L_{p'}(X,E^*)$ converges weak* to $T \in \Pi^L_{p'}(X,E^*)$. Then, for any $x \in X$ and any $v \in E$, $\langle T_\alpha, v m_{x0}\rangle \to \langle T, v m_{x0}\rangle$, i.e. $\langle T_\alpha(x), v\rangle \to \langle T(x), v\rangle$. This means that (T_α) converges to T in the topology of pointwise $\sigma(E^*,E)$-convergence.

Therefore, the identity on $\Pi^L_{p'}(X,E^*)$ is a continuous bijection from the weak* topology to the topology of pointwise $\sigma(E^*,E)$-convergence. On the unit ball, the former is compact and the latter is Hausdorff, so they must coincide. $\square$

In order to answer Question 3 from [FJ09], i.e. identify the dual of the space of Lipschitz p-summing operators from a finite metric space to a Banach space, we will need to "reverse" the duality given by Theorem 4.3. Unsurprisingly, the principle of local reflexivity will play a crucial role.

Lemma 4.4. *When X is a finite metric space, $\Pi^L_p(X,E)^{**}$ and $\Pi^L_p(X,E^{**})$ are (canonically) isometrically isomorphic.*

Proof. As vector spaces, both spaces can be identified with the space of functions from X to E^{**} that vanish at 0, so it will suffice to show equality of their unit balls. By Goldstein's theorem, $B_{\Pi^L_p(X,E)^{**}}$ is the weak*-closure (i.e. the closure under the topology of pointwise $\sigma(E^{**},E^*)$-convergence) of $B_{\Pi^L_p(X,E)}$. Since the Lipschitz p-summing norm does not change if the codomain is enlarged, $B_{\Pi^L_p(X,E)}$ embeds isometrically into $B_{\Pi^L_p(X,E^{**})}$. By Theorem 4.3 the weak*-topology in $\Pi^L_p(X,E^{**})$ (as the dual of $\mathcal{CS}_{p'}(X,E^*)$) is also the topology of $\sigma(E^{**},E^*)$-convergence. Therefore, $B_{\Pi^L_p(X,E)^{**}} \subseteq B_{\Pi^L_p(X,E^{**})}$.

Now fix $T \in \mathrm{Lip}_0(X,E^{**})$. Let F be a finite dimensional subspace of E^{**} containing the span of the image of T such that $F \cap E \neq \{0\}$, and let $\mathcal{A}$ be the directed set of all finite-dimensional subspaces of E^*. Given $\varepsilon \in (0,1)$, by the principle of local reflexivity (say, in the form given in [DJT95, p. 178]) for every $A \in \mathcal{A}$ there exists an injective linear map $u_A : F \to E$ such that: (a) $u_A v = v$ for all $v \in F \cap E$; (b) $\|u_A\| \cdot \left\|u_A^{-1}\right\| \leq 1+\varepsilon$; and (c) $\langle u_A v^{**}, v^* \rangle = \langle v^{**}, v^* \rangle$ for all $v^{**} \in F$ and $v^* \in A$. Note that since $F \cap E$ is not trivial, condition (a) guarantees that $\left\|u_A^{-1}\right\| \geq 1$ and thus $\|u_A\| \leq 1+\varepsilon$ from condition (b). If we set $T_A := u_A \circ T : X \to E$, Then $\pi^L_p(T_A) \leq \|u_A\| \, \pi^L_p(T) \leq (1+\varepsilon)\pi^L_p(T)$ and for every $v^* \in E^*$, since v^* is eventually in $A \in \mathcal{A}$ condition (c) implies that

$$\lim_{A\in\mathcal{A}} \langle T_A x, v^* \rangle = \lim_{A\in\mathcal{A}} \langle u_A T x, v^* \rangle = \langle Tx, v^* \rangle,$$

i.e. the net $(T_A)_{A\in\mathcal{A}}$ converges to T in the topology of pointwise $\sigma(E^{**},E^*)$-convergence. Since $T \in \mathrm{Lip}_0(X,E^{**})$ was arbitrary, this implies that $B_{\Pi^L_p(X,E^{**})}$ is contained in the closure of $(1+\varepsilon)B_{\Pi^L_p(X,E)}$ with respect to the topology of pointwise $\sigma(E^{**},E^*)$-convergence, that is, $B_{\Pi^L_p(X,E^{**})} \subseteq (1+\varepsilon)B_{\Pi^L_p(X,E)^{**}}$. Letting ε go to 0 we conclude that $B_{\Pi^L_p(X,E^{**})} \subseteq B_{\Pi^L_p(X,E)^{**}}$. $\square$

Corollary 4.5. *When X is a finite metric space, $\Pi^L_p(X,E)^* \equiv \mathcal{CS}_{p'}(X,E^*)$.*

Proof. From Theorem 4.3 we have $\mathcal{CS}_{p'}(X,E^*)^* \equiv \Pi^L_p(X,E^{**})$ and Lemma 4.4 gives us $\Pi^L_p(X,E)^{**} \equiv \Pi^L_p(X,E^{**})$, so $\mathcal{CS}_{p'}(X,E^*)^* \equiv \Pi^L_p(X,E)^{**}$. Moreover, the isometry implied in this last inequality is weak*-to-weak* continuous (reasoning as in the proof of Theorem 4.3, weak*-convergence in $\mathcal{CS}_{p'}(X,E^*)^*$ implies pointwise $\sigma(E^{**},E^*)$ convergence, that is, weak*-convergence in $\Pi^L_p(X,E)^{**}$), so it is the adjoint of an isometry between $\mathcal{CS}_{p'}(X,E^*)$ and $\Pi^L_p(X,E)^*$. $\square$

4.4. **An application: a characterization of Lipschitz p-summing operators between metric spaces.** Even though we have been considering only Lipschitz p-summing operators from a metric space into a Banach space, the Chevet-Saphar spaces of molecules can be used to get a new characterization of Lipschitz p-summing operators between metric spaces. Moreover, this characterization has the (potential) advantage of being expressed only in terms of linear operators.

A Lipschitz map $T : X \to Y$ naturally induces a linear map $T_E : \mathcal{M}(X,E) \to \mathcal{M}(Y,E)$ given by

$$T_E\bigg(\sum_{j=1}^n v_j m_{x_j x'_j}\bigg) = \sum_{j=1}^n v_j m_{Tx_j Tx'_j}.$$

First, let us note that $T_E : \mathcal{M}(X,E) \to \mathcal{M}(Y,E)$ is well-defined, i.e. it does not depend on the given representation of a molecule. For that, suppose that a molecule $m : X \to E$ has two representations $\sum_j v_j m_{x_j x'_j}$ and $\sum_i w_i m_{y_i y'_i}$. Then for all $v^* \in E^*$ the real-valued molecule $v^* \circ m$ has representations $\sum_j v^*(v_j) m_{x_j x'_j}$ and $\sum_i v^*(w_i) m_{y_i y'_i}$. Hence, by duality between Æ(X) and $X^\#$ (see Theorem 3.1), for all $f \in X^\#$ we have that

$$\sum_j v^*(v_j)\big[f(x_j) - f(x'_j)\big] = \sum_i v^*(w_i)\big[f(y_i) - f(y'_i)\big].$$

In particular, for any $g \in Y^\#$ we have $g \circ T \in X^\#$ and thus

$$\sum_j v^*(v_j)\big[g(Tx_j) - g(Tx'_j)\big] = \sum_i v^*(w_i)\big[g(Ty_i) - g(Ty'_i)\big],$$

which means that $\sum_j v_j m_{Tx_j x'_j} = \sum_i w_i m_{Ty_i Ty'_i}$ (applying the same arguments in reverse order).

Theorem 4.6. *Let $T : X \to Y$ be a Lipschitz map. The following are equivalent:*
(a) T is Lipschitz p-summing.
(b) For every Banach space E (or only $E = Y^\#$), the operator

$$T_E : \mathcal{CS}_{p'}(X,E) \to \mathcal{CS}_1(Y,E)$$

is continuous.

In this case,

$$\pi_p^L(T) = \Big\|T_{Y^\#} : \mathcal{CS}_{p'}(X,Y^\#) \to \mathcal{CS}_1(Y,Y^\#)\Big\| \geq \big\|T_E : \mathcal{CS}_{p'}(X,E) \to \mathcal{CS}_1(Y,E)\big\|.$$

Proof. Suppose that $T : X \to Y$ is Lipschitz p-summing. Let $\varphi \in \big(\mathcal{CS}_1(Y,E)\big)^*$ with $\|\varphi\| \leq 1$. Since $\big(\mathcal{CS}_1(Y,E)\big)^* \equiv \mathrm{Lip}_0(Y,E^*)$, we can identify φ with a function $L_\varphi \in \mathrm{Lip}_0(Y,E^*)$ with $\mathrm{Lip}(L_\varphi) = \|\varphi\| \leq 1$. Let $m = \sum v_j m_{x_j x'_j} \in \mathcal{M}(X,E)$. Then $T_E(m) = \sum v_j m_{Tx_j Tx'_j}$, so

$$\langle \varphi, T_E(m)\rangle = \sum_j \langle L_\varphi(Tx_j) - L_\varphi(Tx'_j), v_j\rangle = \langle L_\varphi \circ T, m\rangle,$$

and thus

$$\big|\langle \varphi, T_E(m)\rangle\big| = \big|\langle L_\varphi \circ T, m\rangle\big| \leq \pi_p^L(L_\varphi \circ T) cs_{p'}(m) \leq \mathrm{Lip}(L_\varphi)\pi_p^L(T) cs_{p'}(m) \leq \pi_p^L(T) cs_{p'}(m).$$

Taking the supremum over all such φ,

$$cs_1(T_E(m)) \leq \pi_p^L(T) cs_{p'}(m),$$

so $T_E : \mathcal{CS}_{p'}(X,E) \to \mathcal{CS}_1(Y,E)$ is continuous and $\|T_E\| \leq \pi_p^L(T)$.

Now, suppose that $T_{Y^\#} : \mathcal{CS}_{p'}(X,Y^\#) \to \mathcal{CS}_1(Y,Y^\#)$ is continuous and has norm C. Let $j_Y : Y \to (Y^\#)^*$ be the canonical isometric embedding. From the definition of Lipschitz p-summing, it suffices to show that $j_Y \circ T$ is Lipschitz p-summing. Let $m \in \mathcal{M}(X,Y^\#)$. Write $m = \sum_j g_j m_{x_j x'_j}$ with $g_j \in Y^\#$. Then

$$\begin{aligned} \langle j_Y \circ T, m\rangle &= \sum_j \langle j_Y \circ T(x_j) - j_Y \circ (Tx'_j), g_j\rangle = \sum_j \big[g_j(Tx_j) - g_j(Tx'_j)\big] \\ &= \sum_j \langle j_Y, \sum_j g_j m_{Tx_j Tx'_j}\rangle = \langle j_Y, T_{Y^\#}(m)\rangle, \end{aligned}$$

so

$$\big|\langle j_Y \circ T, m\rangle\big| = \big|\langle j_Y, T_{Y^\#}(m)\rangle\big| \leq \mathrm{Lip}(j_Y) cs_1\big(T_{Y^\#}(m)\big) \leq 1 \cdot C cs_{p'}(m).$$

Therefore, from the duality between the p'-Chevet-Saphar norm and the Lipschitz p-summing norm, after taking the supremum over all m with $cs_{p'}(m) \leq 1$ we get

$$\pi_p^L(T) \leq C$$

and the proof is over because, now that we know that T is Lipschitz p-summing, from the first part we get $\pi_p^L(T) \geq C$. $\square$

5. Lapresté norms and Lipschitz (p,r,s)-summing operators

In [Lap76], J.T. Lapresté defined a generalization of the Chevet-Saphar tensor norms. In this section we study the corresponding definition for spaces of molecules.

5.1. **Definition and elementary properties.**

For a molecule $m \in \mathcal{M}(X,E)$, let

$$\mu_{p,r,s}(m) = \inf\left\{ \|(\lambda_j)_j\|_p \, w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j^{-1}, x_j, x'_j)_j\big) w_s\big((\kappa_j v_j)_j\big) \;:\; m = \sum_j v_j m_{x_j x'_j},\ \lambda_j, \kappa_j > 0 \right\}.$$

Recall that for $0 < \beta \leq 1$, a non-negative positively homogeneous functional μ defined on a vector space U is called a *β-seminorm* if $\mu(u_1+u_2)^\beta \leq \mu(u_1)^\beta + \mu(u_2)^\beta$ for all $u_1, u_2 \in U$. If in addition μ vanishes only at 0, it is called a *β-norm*.

Theorem 5.1. *Suppose $1/\beta := 1/p + 1/r + 1/s \geq 1$. Then $\mu_{p,r,s}$ is a β-norm on $\mathcal{M}(X,E)$.*

Proof. It is clear that for any molecule $m \in \mathcal{M}(X,E)$ and any scalar λ, $\mu_{p,r,s}(m) \geq 0$ and $\mu_{p,r,s}(\lambda m) = |\lambda| \mu_{p,r,s}(m)$.

Let $m_1, m_2 \in \mathcal{M}(X,E)$ and $\varepsilon > 0$. Choose a representation $m_1 = \sum_j v_j m_{x_j x'_j}$ and positive reals λ_j, κ_j such that

$$\|(\lambda_j)_j\|_p \, w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j^{-1}, x_j, x'_j)_j\big)\, w_s\big((\kappa_j v_j)_j\big) \leq \mu_{p,r,s}(m_1) + \varepsilon.$$

Multiplying $(\lambda_j)_j$ and $(\kappa_j)_j$ by appropriate positive constants we may in fact assume that

$$\begin{aligned}\|(\lambda_j)_j\|_p &\le \big(\mu_{p,r,s}(m_1)^\beta + \varepsilon\big)^{1/p},\\ w_s\big((\kappa_j v_j)_j\big) &\le \big(\mu_{p,r,s}(m_1)^\beta + \varepsilon\big)^{1/s},\\ w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j^{-1}, x_j, x'_j)_j\big) &\le \big(\mu_{p,r,s}(m_1)^\beta + \varepsilon\big)^{1/r}.\end{aligned}$$

Similarly, choose a representation $m_2 = \sum_i w_i m_{y_i y'_i}$ and positive reals η_i, γ_i such that

$$\begin{aligned}\|(\eta_i)_i\|_p &\le \big(\mu_{p,r,s}(m_2)^\beta + \varepsilon\big)^{1/p},\\ w_s\big((\gamma_i w_i)_j\big) &\le \big(\mu_{p,r,s}(m_2)^\beta + \varepsilon\big)^{1/s},\\ w_r^{\mathrm{Lip}}\big((\gamma_i^{-1}\eta_i^{-1}, y_i, y'_i)_j\big) &\le \big(\mu_{p,r,s}(m_2)^\beta + \varepsilon\big)^{1/r}.\end{aligned}$$

As in the proof of Theorem 4.1, concatenate these representations and accompanying positive reals to get a representation of $m_1 + m_2$ and sequences of positive reals that witness the fact that

$$\mu_{p,r,s}(m_1 + m_2) \le \big(\mu_{p,r,s}(m_1)^\beta + \mu_{p,r,s}(m_2)^\beta + 2\varepsilon\big)^{1/\beta}$$

and hence, letting $\varepsilon \downarrow 0$

$$\mu_{p,r,s}(m_1 + m_2)^\beta \le \mu_{p,r,s}(m_1)^\beta + \mu_{p,r,s}(m_2)^\beta.$$

For a function $T \in \mathrm{Lip}_0(X, E^*)$ that admits a representation as a finite sum of the form $T = \sum_k \tilde\lambda_k v_k^* f_k$ with $\tilde\lambda_k \in \mathbb{R}$, $v_k^* \in E^*$ and $f_k \in X^\#$ (i.e. such that the linearization $\hat T : \text{Æ}(X) \to E^*$ has finite rank) set

$$\theta_{p,r,s}(T) = \inf\left\{ \left\|\big(\tilde\lambda_k\big)_k\right\|_p \big\|\big(\,\|v_k^*\|\,\big)_k\big\|_r \big\|\big(\,\mathrm{Lip}(f_k)\big)_k\big\|_s \right\}$$

where the infimum is taken over all representations of T as above. For any such T and $m = \sum_j v_j m_{x_j x'_j}$, $\lambda_j, \kappa_j > 0$ using the fact that $0 < \beta \le 1$ and Hölder's inequality

$$\begin{aligned}\big|\langle T, m\rangle\big| &= \Big|\sum_{j,k} \tilde\lambda_k \lambda_j v_k^*(v_j)\big[f_k(x_j) - f_k(x'_j)\big]\Big|\\ &\le \sum_{j,k} \big|\tilde\lambda_k \lambda_j \kappa_j v_k^*(v_j)\lambda_j^{-1}\kappa_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big]\big|\\ &\le \Big(\sum_{j,k} \big|\tilde\lambda_k \lambda_j \kappa_j v_k^*(v_j)\lambda_j^{-1}\kappa_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big]\big|^\beta\Big)^{1/\beta}\\ &\le \left\|\big(\tilde\lambda_k\lambda_j\big)_{j,k}\right\|_p \left\|\big(\lambda_j^{-1}\kappa_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big]\big)_{j,k}\right\|_r \left\|\big(\kappa_j v_k^*(v_j)\big)_{j,k}\right\|_s.\end{aligned}$$

Note that

$$\begin{aligned}\left\|\big(\tilde\lambda_k\lambda_j\big)_{j,k}\right\|_p &\le \left\|\big(\tilde\lambda_k\big)_k\right\|_p \|(\lambda_j)_j\|_p\\ \left\|\big(\lambda_j^{-1}\kappa_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big]\big)_{j,k}\right\|_r &\le \big\|\big(\,\mathrm{Lip}(f_k)\big)_k\big\|_r\, w_r^{\mathrm{Lip}}\big((\lambda_j^{-1}, \kappa_j^{-1}, x_j, x'_j)_j\big)\\ \left\|\big(\kappa_j v_k^*(v_j)\big)_{j,k}\right\|_r &\le \big\|\big(\,\|v_k^*\|\,\big)_k\big\|_s\, w_s\big((\kappa_j v_j)_j\big)\end{aligned}$$

so by taking the infimum over all representations of both m and T, and all positive numbers λ_j, κ_j we obtain

$$\big|\langle T, m\rangle\big| \le \mu_{p,r,s}(m)\theta_{p,r,s}(T).$$

Therefore, if $\mu_{p,r,s}(m) = 0$ we have $\langle v^* f, m\rangle = 0$ for all $v^* \in E^*$, $f \in X^\#$. By duality between Æ(X) and $X^\#$, that means the real-valued molecule $v^* \circ m$ is equal to 0 for all $v^* \in E^*$, so we conclude that $m = 0$ and thus $\mu_{p,r,s}$ is a β-norm rather than just a β-seminorm. □

5.2. **Duality.** Just as in the linear case, the dual of the (p, r, s)-Laprest́é norm is the (p', r, s)-summing norm. An operator $T : X \to E$ is called *Lipschitz (p, r, s)-summing* if there is a constant C such that for all $x_j, x'_j \in X$, $v_j \in E^*$, and $\lambda_j, \kappa_j > 0$ we have

$$\Big\|\big(\lambda_j\langle v_j, Tx_j - Tx'_j\rangle\big)_j\Big\|_p \le C w_r^{\mathrm{Lip}}\big((\lambda_j\kappa_j^{-1}, x_j x'_j)_j\big) w_s\big((\kappa_j v_j)\big) \tag{5.1}$$

The smallest such constant C will be denoted by $\pi^L_{p,r,s}(T)$, and $\Pi^L_{p,r,s}(X, E)$ will denote the set of all such operators. A few remarks about this definition are in order. First, when $E = F^*$ it suffices to consider only $v_j \in F$. Also, the case (p, p, ∞) corresponds to Lipschitz p-summing operators from X to E as in [FJ09], whereas the case (q, p, ∞) corresponds to the Lipschitz (q, p)-summing operators from X to E as in [JS09]. Moreover, by the same arguments as in [FJ09], we may take $\lambda_j = 1$ for all j in (5.1). Finally, it is easy to check that $\big(\Pi^L_{p,r,s}(X, E), \pi^L_{p,r,s}\big)$ is a normed space.

Theorem 5.2. *The spaces $M_{p,r,s}(X, E)^*$ and $\Pi^L_{p',r,s}(X, E^*)$ are isometrically isomorphic via the canonical pairing. Moreover, on the unit ball of $\Pi^L_{p',r,s}(X, E^*)$ the weak* topology coincides with the topology of pointwise $\sigma(E^*, E)$-convergence.*

Proof. First, let $T \in \Pi^L_{p',r,s}(X, E^*)$. Then, for any $m = \sum_j v_j m_{x_j x'_j} \in \mathcal{M}(X, E)$ and $\lambda_j, \kappa_j > 0$, by the pairing formula (3.2) and Hölder's inequality

$$\begin{aligned}
\big|\langle T, m\rangle\big| &= \Big|\sum_j \langle Tx_j - Tx'_j, v_j\rangle\Big| \le \sum_j \big|\langle Tx_j - Tx'_j, v_j\rangle\big| \\
&\le \|(\lambda_j)_j\|_p \Big\|\big(\lambda_j^{-1}\langle Tx_j - Tx'_j, v_j\rangle\big)_j\Big\|_{p'} \\
&\le \|(\lambda_j)_j\|_p\, \pi^L_{p',r,s}(T) w_r^{\mathrm{Lip}}\big((\lambda_j^{-1}\kappa_j^{-1}, x_j, x'_j)_j\big) w_s((\kappa_j v_j)_j)
\end{aligned}$$

Taking the infimum over all representations of m and $\lambda_j, \kappa_j > 0$ we conclude that $\big|\langle T, m\rangle\big| \le \pi^L_{p',r,s}(T)\mu_{p,r,s}(m)$. Conversely, let $\varphi \in M_{p,r,s}(X, E)^*$ with $\|\varphi\| = C$, so we have $|\varphi(m)| \le C\mu_{p,r,s}(m)$ for any $m \in \mathcal{M}(X, E)$. Note that φ can be identified with a mapping $T : X \mapsto E^*$ via the formula $\langle Tx, v\rangle = \varphi(vm_{x0})$. Indeed, for $x \in X$ and $v \in E$,

$$\begin{aligned}
|\langle Tx, v\rangle| = |\langle \varphi, vm_{x0}\rangle| &\le C\mu_{p,r,s}(vm_{x0}) \\
&\le C \sup_{v^* \in B_{E^*}} |v^*(v)| \sup_{f \in B_{X^\#}} |f(x) - f(0)| = C\,\|v\|\, d(x, 0)
\end{aligned}$$

so $Tx \in E^*$. Now, suppose $x_j, x'_j \in X$, $\lambda_j, \kappa_j > 0$. For any sequence $(\alpha_j)_j$ of real numbers with $\|(\alpha_j)_j\|_p \le 1$,

$$\begin{aligned}\Big|\sum_j \alpha_j \lambda_j \langle Tx_j - Tx'_j, v_j\rangle\Big| &= \Big|\langle T, \sum_j \alpha_j\lambda_j v_j m_{x_j x'_j}\rangle\Big| \\ &\le C\mu_{p,r,s}\Big(\sum_j \alpha_j\lambda_j v_j m_{x_j x'_j}\Big) \\ &\le C\,\|(\alpha_j)_j\|_p\, w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j^{-1}, x_j, x'_j)_j\big) w_s((\kappa_j v_j)_j) \\ &\le C w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j^{-1}, x_j, x'_j)_j\big) w_s((\kappa_j v_j)_j).\end{aligned}$$

Taking the supremum over all such α,

$$\Big\|\big(\lambda_j\langle Tx_j - Tx'_j, v_j\rangle\big)_j\Big\|_{p'} \le C w_r^{\mathrm{Lip}}\big(\kappa^{-1}\lambda^{-1}, x\big) w_s(\kappa v),$$

i.e. T is Lipschitz (p', r, s)-summing with $\pi^L_{p',r,s}(T) \le C$. For the statement about the weak*-topology, we use the exact same argument as in the proof of Theorem 4.3. □

5.3. **A special case.** Just as in the linear case, when $1/p + 1/r + 1/s = 1$ the dual of $M_{p,r,s}(X,E)$ has another interesting characterization in terms of summing operators. We will make use of the following elementary identity.

Lemma 5.3. *Suppose $1 \le p, r, s < \infty$ and $1 = 1/p + 1/r + 1/s$. Then For $a, b, c \ge 0$,*

$$abc = \inf_{\lambda,\kappa>0}\left\{\frac{\lambda^p}{p}a^p + \frac{\kappa^s}{s}b^s + \frac{\lambda^{-r}\kappa^{-r}}{r}c^r\right\}.$$

Proof. It is an easy calculus exercise to show that

$$a(bc) = \inf_{\lambda>0}\left\{\frac{\lambda^p}{p}a^p + \frac{\lambda^{-p'}}{p'}(bc)^{p'}\right\}.$$

Applying the same idea again to the product bc we get the result. □

The following theorem identifies the dual of $M_{p,r,s}(X,E)$ in this special case.

Theorem 5.4 (**Domination/Factorization**)**.** *Suppose $1/p + 1/r + 1/s = 1$ and let $T \in \mathrm{Lip}_0(X, E^*)$, $C > 0$. The following are equivalent:*

(a) $|\langle T, m\rangle| \le C\mu_{p,r,s}(m)$ for all $m \in \mathcal{M}(X,E)$.

(b) There exist regular Borel probability measures μ and ν on the weak*-compact unit balls $B_{X^\#}$, B_{E^*} (considering $X^\# = \mathscr{F}(X)^*$) such that for all $x, x' \in X$ and $v \in E$,

$$|\langle Tx - Tx', v\rangle| \le C\left[\int_{B_{X^\#}} \big|f(x) - f(x')\big|^r\, d\mu(f)\right]^{1/r}\left[\int_{B_{E^*}} |v^*(v)|^s\, d\nu(v^*)\right]^{1/s}.$$

(c) There exist a Banach space Z, a Lipschitz r-summing operator $R : X \to Z^*$ and a linear s-summing operator $S : E \to Z$ such that $\pi^L_r(R)\cdot\pi_s(S) \le C$ and

$$\langle Tx, v\rangle = \langle Rx, Sv\rangle \qquad \text{for all } x \in X, v \in E;$$

that is, $T = S^* \circ R$.

Note that condition (c) can be considered as a Lipschitz version of (linear) (r,s)-dominated operators, i.e. those that can be factored as a composition of an r-summing operator and the adjoint of an s-summing operator (see, e.g. [DF93, p. 241]).

Proof. We will assume $p,r,s<\infty$ for the sake of simplicity; the other cases have similar proofs (for instance, the case $s=\infty$ follows from the domination theorem for Lipschitz p-summing operators [FJ09, Thm. 1] and Theorem 5.2).

(a) $\Rightarrow$ (b) Consider a molecule $m=\sum_j v_j m_{x_j x_j'}$, $x_j,x_j'\in X$, $v_j\in E$. By definition of $\mu_{p,r,s}$, for any $\lambda,\kappa>0$

$$\Big|\sum_j \langle Tx_j-Ty_j,v_j\rangle\Big|\le$$
$$C\Big(\sum_j \lambda_j^p\Big)^{1/p}\sup_{f\in B_X^{\#}}\Big(\lambda_j^{-r}\kappa_j^{-r}\big|f(x_j)-f(x_j')\big|^r\Big)^{1/r}\sup_{v^*\in B_{E^*}}\Big(\sum_j \kappa_j^s|v^*(v_j)|^s\Big)^{1/s}$$

Applying Lemma 5.3 gives for any $\gamma,\delta>0$,

$$\Big|\sum_j \langle Tx_j-Ty_j,v_j\rangle\Big|\le$$
$$C\sup_{f\in B_{X^{\#}},v^*\in B_{E^*}}\sum_j\Big[\frac{\gamma^p}{p}\lambda_j^p+\frac{\delta^s}{s}\kappa_j^s|v^*(v_j)|^s+\frac{\gamma^{-r}\delta^{-r}}{r}\lambda_j^{-r}\kappa_j^{-r}\big|f(x_j)-f(x_j')\big|^r\Big].$$

This means, after renaming variables, that for all $\lambda_j,\kappa_j>0$

$$\Big|\sum_j \langle Tx_j-Ty_j,v_j\rangle\Big|\le \tag{5.2}$$
$$C\sup_{f\in B_{X^{\#}},v^*\in B_{E^*}}\sum_j\Big[\frac{\lambda_j^p}{p}+\frac{\kappa_j^s}{s}|v^*(v_j)|^s+\frac{\lambda_j^{-r}\kappa_j^{-r}}{r}\big|f(x_j)-f(x_j')\big|^r\Big].$$

We now use the same idea as in the proof of the Pietsch Domination Theorem to find the measures μ and ν. Working on the space $C(B_{X^{\#}}\times B_{E^*})$, consider the set L consisting of functions of the form

$$g_A(f,v^*)=\Big|\sum_{(x,x',v,\lambda,\kappa)\in A}\langle Tx-Tx',v\rangle\Big|-$$
$$C\sum_{(x,x',v,\lambda,\kappa)\in A}\Big[\frac{\lambda^p}{p}+\frac{\kappa^s}{s}|v^*(v_j)|^s+\frac{\lambda^{-r}\kappa^{-r}}{r}\big|f(x_j)-f(x_j')\big|^r\Big],$$

where A is a finite subset of $X\times X\times E\times\mathbb{R}^+\times\mathbb{R}^+$. Then L is a convex set and every function in L takes at least one non-positive value by (5.2). In particular, L is disjoint from the open positive cone P of $C(B_{X^{\#}}\times B_{E^*})$, and hence there exists a regular (finite) Borel measure μ_0 on $B_{X^{\#}}\times B_{E^*}$ that separates L and P. Arguing as usual, we may assume that μ_0 is a probability measure and $\langle f_A, \mu_0\rangle \leq 0$ for every $f_A \in L$. Taking a singleton $A = \{(x, x', v, \lambda, \kappa)\}$ we get

$$
\begin{aligned}
&|\langle Tx - Tx', v\rangle| \leq \\
&\qquad C\int_{B_{X^\#}\times B_{E^*}} \left[\frac{\lambda^p}{p} + \frac{\kappa^s}{s}|v^*(v)|^s + \frac{\lambda^{-r}\kappa^{-r}}{r}\big|f(x) - f(x')\big|^r\right] d\mu_0(f, v^*) \\
&= C\left[\frac{\lambda^p}{p} + \frac{\kappa^s}{s}\int_{B_{X^\#}\times B_{E^*}} |v^*(v)|^s\, d\mu_0(f, v^*) + \frac{\lambda^{-r}\kappa^{-r}}{r}\int_{B_{X^\#}\times B_{E^*}} \big|f(x) - f(x')\big|^r\, d\mu_0(f, v^*)\right].
\end{aligned}
$$

Another application of Lemma 5.3 gives

$$
|\langle Tx - Tx', v\rangle| \leq C\left[\int_{B_{E^*}} |v^*(v)|^s\, d\nu(v^*)\right]^{1/s} \left[\int_{B_{X^\#}} \big|f(x) - f(x')\big|^r\, d\mu(f)\right]^{1/r}
$$

where μ and ν are the marginals of μ_0.

$(b) \Rightarrow (c)$ Let $j_X : X \to L_r(\mu)$ and $j_E : E \to L_s(\nu)$ be given by

$$
(j_X x)(f) = f(x), \qquad (j_E v)(v^*) = v^*(v) \qquad \text{for all } x \in X, v \in E, f \in B_{X^\#}, v^* \in B_{E^*}.
$$

Note that j_X is Lipschitz r-summing (resp. j_E is linear s-summing) since it factors through the canonical injection $C(B_{X^\#}) \to L_r(\mu)$ (resp. through $C(B_{E^*}) \to L_s(\nu)$) and moreover $\pi_r^L(j_X) \leq 1$ (resp. $\pi_s(j_E) \leq 1$).

Let $\tilde{X} := j_X(X) \subset L_r(X)$ and $Z := \overline{j_E(E)} \subset L_s(\nu)$. Define $U : \tilde{X} \to Z^*$ by

$$
\langle U j_X(x), j_E(v)\rangle = \langle Tx, v\rangle \qquad \text{for all } x \in X, v \in E.
$$

First note that this indeed defines an element of Z^*, since by condition (b) we have for all $x \in X$ and $v \in E$

$$
\big|\langle U j_X(x), j_E(v)\rangle\big| = |\langle Tx, v\rangle| \leq C\, \|j_X(x)\|_{L_r(\mu)}\, \|j_E(v)\|_{L_s(\nu)} = C\, \|j_X(x)\|_{L_r(\mu)}\, \|j_E(v)\|_{Z}
$$

and then we extend to all of Z by continuity. Moreover, U is Lipschitz with $\mathrm{Lip}(U) \leq C$: for any $x, x' \in X$, by definition of U and condition (b)

$$
\begin{aligned}
\big\|U j_X(x) - U j_X(x')\big\|_{Z^*} &= \sup_{\|j_E(v)\|_{L_s(\nu)}\leq 1} |\langle U j_X(x) - U j_X(x'), j_E(v)\rangle| \\
&= \sup_{\|j_E(v)\|_{L_s(\nu)}\leq 1} \big|\langle Tx - Tx', v\rangle\big| \\
&\leq \sup_{\|j_E(v)\|_{L_s(\mu)}\leq 1} C\, \big\|j_X(x) - j_X(x')\big\|_{L_r(\mu)}\, \|j_E(v)\|_{L_s(\mu)} \\
&= C\, \|j_X(x) - j_X(y)\|_{L_r(\mu)}\,.
\end{aligned}
$$

Therefore, we have (c) with $S = j_E : E \to Z$ and $R = U j_X : X \to Z^*$, since clearly $\langle Tx, v\rangle = \langle Rx, Sv\rangle$, and

$$
\pi_r^L(R)\pi_s(S) = \pi_r^L(U j_X)\pi_s(S) \leq \mathrm{Lip}(U)\pi_r^L(j_X)\pi_s(j_E) \leq C \cdot 1 \cdot 1 = C.
$$

(c) $\Rightarrow$ (a) Suppose there exist operators R and S as in (c). Then for any molecule $m = \sum_j v_j m_{x_j x'_j}$ and any $\lambda_j, \kappa_j > 0$ the pairing formula 3.2 and Hölder's inequality give

$$\begin{aligned}
\big|\langle T, m\rangle\big| &= \Big|\sum_j \langle Sv_j, Rx_j - Rx'_j\rangle\Big| \le \sum_j \big|\langle Sv_j, Rx_j - Rx'_j\rangle\big| \\
&\le \sum_j \|Sv_j\| \cdot \big\|Rx_j - Rx'_j\big\| = \sum_j \lambda_j\kappa_j \|Sv_j\| \cdot \lambda_j^{-1}\kappa_j^{-1} \big\|Rx_j - Rx'_j\big\| \\
&\le \Big(\sum_j \lambda_j^p\Big)^{1/p} \Big(\sum_j \kappa_j^s \|Sv_j\|^s\Big)^{1/s} \Big(\sum_j \lambda_j^{-r}\kappa_j^{-r} \big\|Rx_j - Rx'_j\big\|^r\Big)^{1/r}.
\end{aligned}$$

Since R is Lipschitz r-summing and S is s-summing, the last expression is at most

$$\pi_r^L(R)\pi_s(S)\Big(\sum_j \lambda_j^p\Big)^{1/p} \sup_{v^*\in B_{E^*}} \Big(\sum_j \kappa_j^s |v^*(v_j)|^s\Big)^{1/s} \sup_{f\in B_X^\#} \Big(\sum_j \lambda_j^{-r}\kappa_j^{-r}\big|f(x_j)-f(x'_j)\big|^r\Big)^{1/r}.$$

Taking the infimum over all representations of m and all $\lambda_j, \kappa_j > 0$, we conclude that $\big|\langle T, m\rangle\big| \le C\mu_{p,r,s}(m)$. □

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